PERGAMON

International Journal of Solids and Structures 39 (2002) 2343–2358

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

On the constitutive modeling of biological soft connective tissues
A general theoretical framework and explicit forms of the tensors of elasticity for strongly anisotropic continuum fiber-reinforced composites at finite strain

Georges Limbert [a,b,*], Mark Taylor [b]

[a] *Biomechanics Research Unit, The Cardiff Medicentre, UWCM, CF14 4UJ Cardiff, UK*

[b] *Bioengineering Sciences Research Group, School of Engineering Sciences, University of Southampton, Highfield, SO17 1BJ Southampton, UK*

Received 1 November 2000; received in revised form 20 December 2001

Abstract

This research describes a *general* theoretical framework for the constitutive modeling of biological soft connective tissues. The approach is based on the theory of continuum fiber-reinforced composites at finite strain. Explicit expressions of the stress tensors in the material and spatial configurations are first established in the general case, without precluding any assumption regarding possible kinematic constraints or any particular mechanical symmetry of the material. Original expressions of the elasticity tensors in the material and spatial configurations are derived and new coupling terms, characterizing the interactions between the constituents of the continuum composite material, are isolated and their biological significance highlighted. Further to this, expressions of the elasticity tensors are degenerated in order to take into account special type of material symmetries. Kinematic constraints and constitutive requirements are also briefly discussed. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Hyperelasticity; Anisotropy; Fiber-reinforced composite; Elasticity tensor; Biological soft tissue; Ligament; Tendon

1. Introduction

Biological tissues are by essence highly complex systems that host chemical, cellular, electrical and mechanical processes. Computational models have the ability to allow investigation into the complexity of these systems and avoid the difficulties in costly experimental studies. However, numerical models require the knowledge of accurate constitutive equations for modeling biological materials. Soft tissue mechanics,

* Corresponding author. Address: Biomechanics Research Unit, The Cardiff Medicentre, UWCM, CF14 4UJ Cardiff, UK. Fax: +44-29-2068-2161.

E-mail address: limbertg@cardiff.ac.uk (G. Limbert).

0020-7683/02/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved.
PII: S0020-7683(02)00084-7

particularly mechanics of ligaments and tendons, is still a vast area for the development of constitutive theories. The main characteristics of biological soft connective tissues is that they sustain large deformations and displacements, have a highly nonlinear behavior and possess strongly anisotropic mechanical properties (Fung, 1981).

Ligaments and tendons are dense connective tissues consisting primarily of parallel-fibered collagenous tissues embedded in a solid matrix. These tissues are composed largely of *collagen*, a fibrous protein constituting approximately one third of the total protein mass in the human body (White et al., 1964). Like other connective tissues, ligaments consist of relatively few cells (fibroblasts) and an abundant extracellular matrix. In general, the cellular material occupies about 20% of the total tissue volume, while the extracellular matrix accounts for the remaining 80%. About 70% of the matrix, also known as *ground substance* consists of water and approximately 30% as solids. These solids are *collagen*, *proteoglycans*, a small amount of *elastin* and other *glycoproteins* such as *actin* and *fibronectin* (Frank and Shrive, 1999). Roughly, 70–80% of the dry weight of normal tendon or ligament is composed of Type I collagen, also found in skin and bone (Fung, 1981).

Tendons are subjected to high unidirectional tensile loads and therefore their (large) collagen fibers are aligned in an orderly parallel arrangement. Ligaments are mainly subjected to uniaxial tensile loads but can also undergo mechanical actions in other nonpreferred directions and more complex loading conditions (shear, contact interactions with bony structures). In consequence, and according to their physiological role, their fibers are not necessarily completely parallel but can form a complex network of interlaced fibers leading to strongly anisotropic mechanical properties (Amiel et al., 1982; Kennedy et al., 1976). The collateral ligaments of the knee are made of collagen fibers mostly parallel whereas the cruciate ligaments are composed of more interwoven fibers. The fibrous architecture and properties are also dependent on the specific location within a ligament or a tendon, namely when one looks at the insertion into bone (Woo et al., 1988).

Due to their structural properties ligaments can be considered as composite materials where one or several families of fibers, namely collagen fibers, are embedded in a highly compliant solid matrix (i.e. the ground substance made of proteoglycans, water, collagen and glycoproteins).

The simplest case of anisotropic material is represented by an isotropic solid matrix containing one family of fibers possessing a single preferred principal direction (at least, locally). This represents *transversely isotropic* symmetry. This formulation is suitable to describe the constitutive behavior of tendons and ligaments possessing mostly parallel collagen fibers. Weiss et al. (1996) used successfully this approach to describe and simulate the behavior of fascia lata tendons and the medial collateral ligament. However, when the soft connective tissue considered is made with branching and interwoven collagen fibers (like the cruciate ligaments) that give rise to strongly anisotropic mechanical properties, it can prove relevant to consider two distinct families of fibers (Hirokawa and Tsuruno, 2000).

The first objective of the present paper is to describe a general theoretical framework suitable for the constitutive modeling of biological soft connective tissues such as ligaments and tendons and presenting it in the most self-contained format as possible. This is achieved by looking at the necessary definitions, theorems and constitutive requirements used in the formulation of an objective constitutive law. The approach adopted here is based on the seminal work of Spencer (1992). The basic idea is to provide a global description of the composite structure at the continuum level by postulating the existence of a strain energy function dependent on strain invariants and structural tensors from which the stress and elasticity tensors are derived. This approach has been successfully used by various authors, namely Hirokawa and Tsuruno (2000), Weiss et al. (1996) for finite element modeling of ligaments and tendons and by Holzapfel et al. (1996), Humphrey (1990) and Humphrey and Yin (1987) for finite element modeling of cardiac tissue mechanics.

The second objective of our research is to extend the developments of Spencer (1992) by providing entirely new explicit expressions for the elasticity tensors in the spatial and material descriptions in the most general case, that is, when no assumption is made regarding the particular orientation of any of the two families of fibers or regarding any simplifying kinematics hypothesis such as incompressibility or inextensibility.

To the best of our knowledge, this aspect is missing in the relevant literature. We believe that the full generality attached to the terms of the elasticity tensor can be helpful in exploring and incorporating into the constitutive formulation complex interactions between the components of the fiber-reinforced composite material that can be missed otherwise. Moreover, elasticity tensors are essential in investigating mathematical properties of the constitutive laws and are a prerequisite in any incremental type nonlinear finite element method.

2. Basic results in continuum mechanics

Before developing our constitutive model the basic notations and results relevant to the formulation of anisotropic hyperelasticity are given below.

2.1. Kinematics

Let B be a continuum body which is a set of points, referred to as particles. Lets assume that there exists a one-to-one mapping, called a *configuration* of B, $\chi : \mathsf{B} \to \mathscr{E} = \Re^3$, twice continuously differentiable (as its inverse χ^{-1}) which puts into correspondence B with some region, referred as $\mathscr{B}$, of the Euclidean point space $\mathscr{E} = \Re^3$. Let be: $\mathscr{B}_0 = \chi_0(\mathsf{B}) \subset \Re^3$ and $\mathscr{B} = \chi(\mathsf{B}) \subset \Re^3$, respectively the *reference* and *current* positions of B. A point P of B is labeled $\mathbf{X} = \chi_0(\mathsf{B})$ in $\mathscr{B}_0$ and $\mathbf{x} = \chi(\mathsf{B})$ in $\mathscr{B}$. The one-to-one mapping $\varphi = \chi \circ \chi^{-1} \subset \Re^3$ is the deformation from $\mathscr{B}_0$ to $\mathscr{B}$. Upon deformation, the material point $P(\mathbf{X})$ is mapped into a spatial position $P'(\mathbf{x})$ by means of φ where $\mathbf{u}$, the displacement field, is introduced:

$$\mathbf{x} = \varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}) \tag{1}$$

The deformation gradient $\mathbf{F}$ is the derivative of the deformation, using the notation:

$$\mathbf{F}(\mathbf{X}) := \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{X}} = \frac{\partial \varphi}{\partial \mathbf{X}} = \sum_{i,I=1}^{3} \frac{\partial \varphi_i}{\partial X_I} \mathbf{e_i} \otimes \mathbf{E_I} \tag{2}$$

$\{\mathbf{E_I}\}_{\mathrm{I}=1,2,3}$ and $\{\mathbf{e_i}\}_{\mathrm{i}=1,2,3}$ are fixed orthonormal bases in $\mathscr{B}_0$ and $\mathscr{B}$ respectively. The uppercase and lowercase letters used in indicial notation refer to the reference and the deformed (current) configuration respectively. The local condition of impenetrability of matter requires that $J(\mathbf{X}) = \det[\mathbf{F}(\mathbf{X})] = \rho(\mathbf{x})/\rho_0(\mathbf{X}) > 0$ where "det" represents the determinant of the linear transformation $[\mathbf{F}]$ and, ρ_0 and ρ, are the density of the material, respectively in the reference and deformed configurations. Following standard usage, one denotes $\mathscr{L}(\Re^3, \Re^3)$ the vector space of linear transformations in $\Re^3$. Then $\mathscr{L}^+$ is defined as

$$\mathscr{L}^+ := \left\{\mathbf{T} \in \mathscr{L}(\Re^3, \Re^3) / \det(\mathbf{T}) > 0\right\} \tag{3}$$

For fixed $\mathbf{X} \in \mathscr{B}_0$, $\mathbf{T}(\mathbf{X}) \subset \mathscr{L}^+$, one also defines $\mathscr{S}^+$ and $\mathscr{O}^+$:

$$\mathscr{S}^+ := \left\{\mathbf{T} \in \mathscr{L}^+ / \mathbf{T}^{\mathrm{T}} = \mathbf{T}\right\} \tag{4}$$

$$\mathscr{O}^+ := \left\{\mathbf{T} \in \mathscr{L}^+ / \mathbf{T}^{\mathrm{T}} \cdot \mathbf{T} = \mathbf{1}\right\} \tag{5}$$

where the superscript "T" denotes the transpose of the linear transformation, and $\mathbf{1}$ is the second-order identity tensor. The right and left Cauchy–Green deformation tensors are respectively defined as

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} \quad \text{and} \quad \mathbf{b} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} \tag{6}$$

Under conditions of regularity for $\mathbf{C}$ and $\mathbf{b}$, $\mathbf{U}$ and $\mathbf{V}$, the right and left stretch tensors, can be defined as unique, symmetric, positive-definite square roots of $\mathbf{C}$ and $\mathbf{b}$, respectively. From the *polar decomposition theorem* (Mardsen and Hughes, 1994), if φ is regular enough, it can be stated that: for each $\mathbf{X} \in \mathscr{B}_0$ there exists an orthogonal transformation $\mathbf{R}(\mathbf{X}) : \mathscr{O}^+ \rightarrow \mathfrak{R}^3$ such that:

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R} \tag{7}$$

2.2. Finite elasticity

Materials for which the constitutive behavior depends only on the current state of deformation are called *elastic* (Ogden, 1984). Furthermore, if the work done by the stresses during a deformation process is path-independent the material is said to be *hyperelastic*. As a consequence of the path-independent behavior and the fact that the first Piola–Kirchhoff stress tensor $\mathbf{P}$ is work conjugate with the rate of deformation gradient $\dot{\mathbf{F}}$, a *stored strain energy function* or *elastic potential per unit undeformed volume W*, $W = W(\mathbf{X}, \mathbf{F}(\mathbf{X})) : \mathscr{B}_0 \times \mathscr{L}^+ \rightarrow \mathfrak{R}$, can be established as the work done by the stresses from the initial to the current position (time t_0 to time t). In the case of a nondissipative process (which corresponds to the hypothesis of the present developments), W correspond to a *Helmholtz free energy of deformation function* $\overline{\Psi}$. The work defined above is expressed as

$$W[\mathbf{X}, \mathbf{F}(\mathbf{X})] = \overline{\Psi}[\mathbf{X}, \mathbf{F}(\mathbf{X})] = \int_{t_0}^{t} \mathbf{P}[\mathbf{X}, \mathbf{F}(\mathbf{X})] : \dot{\mathbf{F}} \, \mathrm{d}t \tag{8}$$

To define an objective constitutive law, $\overline{\Psi}$ must satisfy the *principle of objectivity* or *material frame indifference* (Mardsen and Hughes, 1994) which is expressed in mathematical terms as follows:

$$\overline{\Psi}(\mathbf{X}, \mathbf{F}) = \overline{\Psi}(\mathbf{X}, \mathbf{Q} \cdot \mathbf{F}) \quad \forall \; (\mathbf{F}, \mathbf{Q}) \in \mathscr{L}^+ \times \mathscr{O}^+ \tag{9}$$

This means that the dependence of constitutive relations on $\mathbf{F}$ must only come through the part of $\mathbf{F}$ causing stretching, that is $\mathbf{U}$. Thus, frame indifference is equivalent to the assertion that $\overline{\Psi}$ depends on $\mathbf{F}$ trough $\mathbf{C}$. In summary, it exists a function $\Psi : \mathscr{B}_0 \times \mathscr{S}^+ \rightarrow \mathfrak{R}$ such that:

$$\overline{\Psi}(\mathbf{X}, \mathbf{F}) = \Psi(\mathbf{X}, \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}) \equiv \Psi(\mathbf{X}, \mathbf{C}) \quad \forall \; \mathbf{C} \in \mathscr{S}^+ \tag{10}$$

2.3. Material symmetry

Extensive work has been done on the subject of material symmetry (Cohen and Wang, 1987; Coleman and Noll, 1964; Ericksen, 1978, 1979; Ericksen and Rivlin, 1954; Negahban and Wineman, 1989a, 1989b; Wineman and Pipkin, 1964; Zheng and Boeheler, 1994). Before going further it is relevant to refer to an important principle, namely the *Neumann's principle* (Hahn, 1987), which states that:

> The symmetry group of a given material must be included in the symmetry group of any tensor function in any constitutive laws of the material.

Boehler (1978) demonstrated that any scalar-, vector-, and second-order tensor-valued functions of vectors and second-order tensors relative to any anisotropy characterized in terms of vectors and second-order tensors can be expressed as an isotropic function of the original tensor agencies and the structural tensors as additional agencies. This means that the strain energy function of an anisotropic material can be expressed as an isotropic function of its classical three principal strain invariants (as in the isotropic case) plus invariants relating the right Cauchy–Green deformation tensor and any combination of structural tensors characterizing the anisotropy. Material symmetries are characterized by symmetry groups that impose restrictions on the form of the strain energy function (Ogden, 1984). Any orthogonal transformation member of the symmetry group of the material will leave the strain energy function unchanged when applied to the material in the natural state (prior to deformation).

3. Fiber-reinforced continuum

To describe the constitutive behavior of biological soft connective tissue in the most general case, we consider a material constructed from two families of fibers continuously distributed in a (highly) compliant solid isotropic matrix (Fig. 1). The result of the geometrical and mechanical interactions of the three constituents gives the material strongly anisotropic macroscopic properties (Spencer, 1992). The two family of fibers F_1 and F_2 are characterized by, respectively a unit vector $\mathbf{n}_0(\mathbf{X})$ and a unit vector $\mathbf{m}_0(\mathbf{X})$, both defined in the reference configuration $\mathscr{B}_0$. These two vectors define locally the preferred directions from which the anisotropy directly arises and then the fiber directions can vary within the material. For sake of clarity, the possible dependence on $\mathbf{X}$ of $\mathbf{n}_0$ and $\mathbf{m}_0$ will be omitted in the next developments. The existence of a strain energy function $\overline{\Psi}$, isotropic function of its arguments, is postulated. The strain energy function $\overline{\Psi}$ is only a function of $\mathbf{X}$, $\mathbf{C}$, $\mathbf{n}_0$ and $\mathbf{m}_0$ and is therefore written as $\overline{\Psi} = \overline{\Psi}(\mathbf{X}, \mathbf{C}, \mathbf{n}_0, \mathbf{m}_0)$. The two structural tensors $\mathbf{N}_0 := \mathbf{n}_0 \otimes \mathbf{n}_0$ and $\mathbf{M}_0 := \mathbf{m}_0 \otimes \mathbf{m}_0$ have to be introduced. They reflect the local structural arrangement of the fibers and thus define local directional properties of the composite material.

This theoretical aspect connects to the fact that connective tissues have different structural properties according to the location (Frank and Shrive, 1999) and shows that the theory presented can take into account this feature. The invariance requirement of the strain energy function with respect to the material symmetry group can be stated as follows: $\forall\ (\mathbf{X}, \mathbf{Q}, \mathbf{C}) \in \mathscr{B}_0 \times \mathscr{O}^+ \times \mathscr{S}^+$

$$\overline{\Psi}(\mathbf{X}, \mathbf{C}, \mathbf{n}_0 \otimes \mathbf{n}_0, \mathbf{m}_0 \otimes \mathbf{m}_0) = \overline{\Psi}(\mathbf{X}, \mathbf{Q} \cdot \mathbf{C} \cdot \mathbf{Q}^{\mathrm{T}}, \mathbf{Q} \cdot \mathbf{n}_0 \otimes \mathbf{n}_0 \cdot \mathbf{Q}^{\mathrm{T}}, \mathbf{Q} \cdot \mathbf{m}_0 \otimes \mathbf{m}_0 \cdot \mathbf{Q}^{\mathrm{T}}) \tag{11}$$

A set of eight invariants ($I_{\alpha,\,\alpha=1-8}$) are necessary to form the *irreducible integrity bases* of the tensors $\mathbf{C}$, $\mathbf{N}_0$ and $\mathbf{M}_0$ (Spencer, 1992). In other words, it must exist a strain energy function Ψ, $\Psi : \mathfrak{R}^+ \times \mathfrak{R}^+ \times \mathfrak{R}^+ \times \mathfrak{R}^+ \times \mathfrak{R}^+ \times \mathfrak{R}^+ \times \mathfrak{R}^+ \times \mathfrak{R}^+ \to \mathfrak{R}$ such that $\overline{\Psi}$ can be written in the following form:

$$\overline{\Psi}(\mathbf{X}, \mathbf{C}, \mathbf{n}_0, \mathbf{m}_0) = \Psi[\mathbf{X}, I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_4(\mathbf{C}, \mathbf{n}_0), I_5(\mathbf{C}, \mathbf{n}_0), I_6(\mathbf{C}, \mathbf{m}_0), I_7(\mathbf{C}, \mathbf{m}_0), I_8(\mathbf{C}, \mathbf{n}_0, \mathbf{m}_0)] \tag{12}$$

The latest form of the strain energy function automatically satisfies the *principle of frame indifference* because it is defined from quantities associated with the reference state. From here we adopt the same

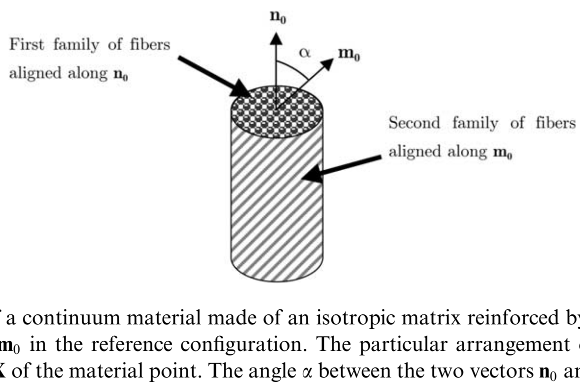

Fig. 1. Simplified representation of a continuum material made of an isotropic matrix reinforced by two families of fibers respectively associated with directions $\mathbf{n}_0$ and $\mathbf{m}_0$ in the reference configuration. The particular arrangement of the fibers is defined locally and therefore depends on the position $\mathbf{X}$ of the material point. The angle α between the two vectors $\mathbf{n}_0$ and $\mathbf{m}_0$ characterizes the local degree of anisotropy.

terminology as used by Spencer (1992) for the tensorial invariants ($I_{\alpha,\,\alpha=1\text{–}8}$). Observing that $I_4 = (\lambda_{\mathbf{n}_0})^2$ and $I_6 = (\lambda_{\mathbf{m}_0})^2$ where $\lambda_{\mathbf{n}_0}$ and $\lambda_{\mathbf{m}_0}$ denote respectively the stretch associated with the direction $\mathbf{n}_0$ and the stretch associated with the direction $\mathbf{m}_0$, allows an easy physical interpretation of the invariants I_4 and I_6. These two invariants are directly related to the type of data one can obtain experimentally when performing tensile tests on a soft tissue specimen. This makes straightforward the parameter identification for any constitutive law using I_4 and/or I_6. Upon deformation, the unit vectors $\mathbf{n}_0$ and $\mathbf{m}_0$ (from the reference configuration) are transformed into a vector $\lambda_{\mathbf{n}_0}\mathbf{n}$ and $\lambda_{\mathbf{m}_0}\mathbf{m}$ respectively where $\mathbf{n}$ and $\mathbf{m}$ represent the unit vectors associated with each family of fibers in the distorted configuration.

By first differentiation of Ψ with respect to $\mathbf{C}$ and push-forward operation of the resulting tensor, the second Piola–Kirchhoff and Cauchy stress tensors, $\mathbf{S}$ and $\boldsymbol{\sigma}$, are respectively obtained as follows:

$$\mathbf{S} = 2\left[(\Psi_1 + I_1\Psi_2)\mathbf{1} - \Psi_2\mathbf{C} + I_3\Psi_3\mathbf{C}^{-1} + \Psi_4\mathbf{N}_0 + \Psi_5\mathbf{N}_{0\mathbf{C}} + \Psi_6\mathbf{M}_0 + \Psi_7\mathbf{M}_{0\mathbf{C}} + \Psi_8\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\right] \tag{13}$$

$$\boldsymbol{\sigma} = \frac{2}{J}\left[(\Psi_1 + I_1\Psi_2)\mathbf{b} - \Psi_2\mathbf{b}^2 + I_3\Psi_3\mathbf{1} + I_4\Psi_4\mathbf{N} + I_4\Psi_5\mathbf{N}_\mathbf{b} + I_6\Psi_6\mathbf{M} + I_6\Psi_7\mathbf{M}_\mathbf{b} + \sqrt{I_4I_6}\,\Psi_8\mathbf{T}_{\mathbf{nm}}\right] \tag{14}$$

where the following notation has been introduced: $\Psi_\alpha := \partial\Psi/\partial I_{\alpha,\,\alpha=1\text{–}8}$ and where:

$$\mathbf{N} := \mathbf{n}\otimes\mathbf{n}, \quad \mathbf{M} := \mathbf{m}\otimes\mathbf{m} \tag{15}$$

$$\mathbf{N}_\mathbf{b} := \mathbf{N}\cdot\mathbf{b} + \mathbf{N}\cdot\mathbf{b}, \quad \mathbf{M}_\mathbf{b} := \mathbf{M}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{M}, \quad \mathbf{T}_{\mathbf{nm}} := \tfrac{1}{2}\mathbf{n}_0\cdot\mathbf{m}_0(\mathbf{n}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{n}) \tag{16}$$

Having reviewed the necessary theoretical background for strongly anisotropic fiber-reinforced composites (Spencer, 1992), we will now go in to develop new *general explicit* expression of the tensors of elasticity (material and spatial versions) by considering *all* the possible mutual interactions between the matrix and the two families of fibers.

4. Definition of the elasticity tensors

Weiss et al. (1996) established closed form expressions of the elasticity tensors (material and spatial configurations) for a continuum composite but reinforced by a single family of fibers and by making the hypothesis of incompressible behavior. This latest assumption excludes all the terms of the strain energy function involving the third invariant I_3 of the Cauchy–Green deformation tensors. In this section, the expressions of the elasticity tensors in the material and the spatial configurations are derived. Not only does the elastic tensor specify the response of a material to applied stresses, but it also gives criteria about the actual stability of the structure. However, the discussion of the later concept is out of the scope of this work.

4.1. Elasticity tensor in the material configuration

The material elasticity tensor is obtained by differentiation of the second Piola–Kirchhoff stress tensor $\mathbf{S}$ with respect to the deformation tensor $\mathbf{C}$, as given in Eq. (17):

$$\mathbf{A}^{m} = 4\frac{\partial^2\Psi}{\partial\mathbf{C}\,\partial\mathbf{C}} = 2\frac{\partial\mathbf{S}}{\partial\mathbf{C}} = 2\frac{\partial S_{IJ}}{\partial C_{KL}}\mathbf{E_I}\otimes\mathbf{E_J}\otimes\mathbf{E_K}\otimes\mathbf{E_L} \tag{17}$$

The elasticity tensor thus defined possesses the so-called *minor* and *major symmetries* which can be expressed in indicial notation:

$$A^{m}_{IJKL} = A^{m}_{KLIJ} = A^{m}_{IJLK} = A^{m}_{JILK} \tag{18}$$

In order to obtain a convenient form for $\mathbf{A}^{m}$, we introduce the following notations (Mardsen and Hughes, 1994):

$$(\mathbf{I})_{IJKL} := \frac{\partial C_{IJ}}{\partial C_{KL}} = \frac{1}{2}\left(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}\right) \tag{19}$$

$$(\mathbf{I}_{\mathbf{C}^{-1}})_{IJKL} := \frac{\partial C^{-1}_{IJ}}{\partial C_{KL}} = -\frac{1}{2}\left(C^{-1}_{IK}C^{-1}_{JL} + C^{-1}_{IL}C^{-1}_{JK}\right) \tag{20}$$

$\mathbf{I}$ is the identity mapping on the six-dimensional space of symmetric second-order tensors and $\boldsymbol{\delta}$ is the Kronecker tensor ($\delta_{IJ} = 1$ if $I = J$, 0 otherwise). Differentiating (13) with respect to $\mathbf{C}$ leads to the following non-reduced form for $\mathbf{A}^{m}$:

$$\begin{aligned}\mathbf{A}^{m} = {} & 4\left[\mathbf{1}\otimes\frac{\partial\Psi_1}{\partial\mathbf{C}} + \Psi_2\mathbf{1}\otimes\frac{\partial I_1}{\partial\mathbf{C}} + I_1\mathbf{1}\otimes\frac{\partial\Psi_2}{\partial\mathbf{C}} - \mathbf{C}\otimes\frac{\partial\Psi_2}{\partial\mathbf{C}} - \Psi_2\frac{\partial\mathbf{C}}{\partial C}\right] \\ & + 4\left[\Psi_3\mathbf{C}^{-1}\otimes\frac{\partial I_3}{\partial\mathbf{C}} + I_3\mathbf{C}^{-1}\otimes\frac{\partial\Psi_3}{\partial\mathbf{C}} + I_3\Psi_3\frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}}\right] + 4\left[\mathbf{N}_0\otimes\frac{\partial\Psi_4}{\partial\mathbf{C}} + \mathbf{N}_{0\mathbf{C}}\otimes\frac{\partial\Psi_5}{\partial\mathbf{C}} + \Psi_5(\mathbf{N}_0\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{N}_0)\right] \\ & + 4\left[\mathbf{M}_0\otimes\frac{\partial\Psi_6}{\partial\mathbf{C}} + \mathbf{M}_{0\mathbf{C}}\otimes\frac{\partial\Psi_7}{\partial\mathbf{C}} + \Psi_7(\mathbf{M}_0\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{M}_0)\right] + 4\left[\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\frac{\partial\Psi_8}{\partial\mathbf{C}}\right]\end{aligned} \tag{21}$$

After introducing the following notation: $\Psi_{\alpha\beta} := \partial^2\Psi/\partial I_\alpha\,\partial I_{\beta,\,\beta=1\text{–}8}$, development of the second derivatives of Ψ, application of the chain rule and lengthily algebraic manipulations, the material elasticity tensor is obtained in the following form:

$$\mathbf{A}^{m} := \bar{\mathbf{A}}^{m}_{m} + \tilde{\mathbf{A}}^{m}_{m} + \bar{\mathbf{A}}^{m}_{F_1m} + \tilde{\mathbf{A}}^{m}_{F_1m} + \bar{\mathbf{A}}^{m}_{F_2m} + \tilde{\mathbf{A}}^{m}_{F_2m} + \bar{\mathbf{A}}^{m}_{F_1F_2m} + \tilde{\mathbf{A}}^{m}_{F_1F_2m} + \mathbf{A}^{m}_{F_1F_2} \tag{22}$$

where $\mathbf{A}^{m}$ has been split into several contributions which characterize specific interactions between the matrix and the fibers and between the two families of fibers (Eqs. (23)–(31)). Table 1 highlights the crossed contributions of the differential terms of the strain energy function with respect to the three constituents of the material.

$$\bar{\mathbf{A}}^{m}_{m} := 4\left[(\Psi_{11} + 2I_1\Psi_{12} + \Psi_2 + I_1^2\Psi_{22})\mathbf{1}\otimes\mathbf{1} - (\Psi_{12} + I_1\Psi_{22})(\mathbf{1}\otimes\mathbf{C} + \mathbf{C}\otimes\mathbf{1})\right] + 4[\Psi_{22}(\mathbf{C}\otimes\mathbf{C}) - \Psi_2\mathbf{I}] \tag{23}$$

The term $\bar{\mathbf{A}}^{m}_{m}$ is made of the isotropic components of the derivatives of the strain energy function with respect to the two first invariants, I_1 and I_2, of the Cauchy–Green deformation tensors. I_1 has an easy

Table 1
Table summarizing the various contributions to the material elasticity tensor by separating the various differential terms of the strain energy function

Interactions	Matrix		Family of fibers F_1		Family of fibers F_2	
Matrix	$\partial(I_1,I_2)$ $\bar{\mathbf{A}}^{m}_{m}$	$\partial(I_1,I_2,I_3)$ $\tilde{\mathbf{A}}^{m}_{m}$	$\partial(I_1,I_2,I_4,I_5)$ $\bar{\mathbf{A}}^{m}_{F_1m}$ $\partial(I_2,I_8)$ $\bar{\mathbf{A}}^{m}_{F_1F_2m}$	$\partial(I_3,I_4,I_5)$ $\tilde{\mathbf{A}}^{m}_{F_1m}$ $\partial(I_3,I_8)$ $\tilde{\mathbf{A}}^{m}_{F_1F_2m}$	$\partial(I_1,I_2,I_6,I_7)$ $\bar{\mathbf{A}}^{m}_{F_2m}$ $\partial(I_2,I_8)$ $\bar{\mathbf{A}}^{m}_{F_1F_2m}$	$\partial(I_3,I_6,I_7)$ $\tilde{\mathbf{A}}^{m}_{F_2m}$ $\partial(I_3,I_8)$ $\tilde{\mathbf{A}}^{m}_{F_1F_2m}$
Family of fibers F_1	$\partial(I_1,I_2,I_4,I_5)$ $\bar{\mathbf{A}}^{m}_{F_1m}$	$\partial(I_3,I_4,I_5)$ $\tilde{\mathbf{A}}^{m}_{F_1m}$	$\partial(I_4,I_5)$ $\bar{\mathbf{A}}^{m}_{F_1m}$		$\partial(I_4,I_5,I_6,I_7,I_8)$ $\mathbf{A}^{m}_{F_1F_2}$	
Family of fibers F_2	$\partial(I_1,I_2,I_6,I_7)$ $\bar{\mathbf{A}}^{m}_{F_2m}$	$\partial(I_3,I_6,I_7)$ $\tilde{\mathbf{A}}^{m}_{F_2m}$	$\partial(I_4,I_5,I_6,I_7,I_8)$ $\mathbf{A}^{m}_{F_1F_2}$		$\partial(I_6,I_7)$ $\bar{\mathbf{A}}^{m}_{F_2m}$	

The symbol "∂", placed before a bracket containing invariants I_α, means that the corresponding term of the elasticity tensor contains partial derivatives of the strain energy function Ψ with respect to the invariant(s) considered.

physical interpretation as it corresponds to the sum of the square of the principal stretches. Uniaxial, shear, biaxial and equibiaxial tension are examples of tests that are performed on a soft tissue sample in order to obtain this information. Compression tests may be required to compute out-of-planes stress if the material is not assumed to be incompressible or simply because different behaviors in compression and tension of the matrix are considered. If Ψ_2 is null, the matrix has a constant shear modulus.

Thus if a variable shear modulus is to be accounted for in the constitutive law, one must define at least function of I_2 of degree one.

Terms containing the derivatives of Ψ with respect to I_3 are isolated (Eq. (24)) in order to highlight the terms of the elasticity tensor that are directly related to change of volume. It is relevant to emphasize that this does not constitute the classic decomposition resulting from the split of the deformation gradient $\mathbf{F}$ into a volumetric and a deviatoric part, as often used in incompressible finite element analyses (Flory, 1961). Our decomposition concerns only the terms of the strain energy function and not those involving $\mathbf{C}$. By examining the expression of $\tilde{\mathbf{A}}^{m}_{m}$, it appears that the hypothesis of incompressibility ($I_3 = 1 \Rightarrow \tilde{\mathbf{A}}^{m}_{m} = 0$) provides a significant simplification at least at the constitutive formulation level, not the experimental one. Soft tissues are very often assumed to have an isochoric behavior because of their high water content. However, due to the improvement of experimental methods it appears that a fluid exudation can be observed when a ligament is subjected to a mechanical loading as reported by Thielke et al. (1995)). In consequence, it seems relevant to consider this aspect by using either using a porohyperelastic formulation or the present formulation that takes into account the corresponding volumetric terms that may generate non negligible coupled actions between the various constituents of the composite material.

The compressibility can be dependent on the state of deformation within the matrix $[(\partial^2\Psi/\partial I_\alpha\,\partial I_3)_{\alpha=1,2} \neq \mathbf{0}]$ and this could be captured by coupled terms of Ψ including the invariants I_1, I_2 and I_3.

$$\begin{aligned}\tilde{\mathbf{A}}^{m}_{m} := {} & 4\big[\big(I_3\Psi_3 + I_3^2\Psi_{33}\big)\mathbf{C}^{-1}\otimes\mathbf{C}^{-1} + I_3\Psi_3\mathbf{I}_{\mathbf{C}^{-1}} + I_3(\Psi_{13} + I_1\Psi_{23})\big(\mathbf{1}\otimes\mathbf{C}^{-1} + \mathbf{C}^{-1}\otimes\mathbf{1}\big)\big] \\ & + 4\big[-I_3\Psi_{23}\big(\mathbf{C}\otimes\mathbf{C}^{-1} + \mathbf{C}^{-1}\otimes\mathbf{C}\big)\big] \end{aligned} \tag{24}$$

Terms $\bar{\mathbf{A}}^{m}_{F_1m}$ and $\bar{\mathbf{A}}^{m}_{F_2m}$ of the elasticity tensor characterize the interactions between the isotropic matrix and, respectively, the families of fibers F_1 and F_2. Some of the effects governed by I_4 and I_5 are probably identical despite the relative independence of I_4 and I_5 as tensorial invariants. Naturally, similar remarks apply to I_6 and I_7. The most obvious kind of interactions between matrix and fibers in soft connective tissues is probably shear but one can imagine more complex interactions by using appropriate coupling functions. The experimental characterization of these combined interactions is probably one of the biggest challenges when developing constitutive laws. Moreover, I_5 and I_7 do not have an immediate physical interpretation and this can be subject to further investigation. Deformation of the matrix can produce elongation of the fibers and the reverse effect can also be envisaged.

As suggested by Minns and Soden (1973), an important function of the collagen fibers is to ensure a uniform distribution of deformation and thus avoiding excessive local deformations which are likely to induce early failure of the soft tissue.

$$\begin{aligned}\bar{\mathbf{A}}^{m}_{F_1m} := {} & 4[(\Psi_{14} + I_1\Psi_{24} + \Psi_5)(\mathbf{1}\otimes\mathbf{N}_0 + \mathbf{N}_0\otimes\mathbf{1}) + (\Psi_{15} + I_1\Psi_{25})(\mathbf{1}\otimes\mathbf{N}_{0\mathbf{C}} + \mathbf{N}_{0\mathbf{C}}\otimes\mathbf{1})] \\ & + 4[-\Psi_{24}(\mathbf{C}\otimes\mathbf{N}_0 + \mathbf{N}_0\otimes\mathbf{C}) - \Psi_{25}(\mathbf{C}\otimes\mathbf{N}_{0\mathbf{C}} + \mathbf{N}_{0\mathbf{C}}\otimes\mathbf{C})] \\ & + 4[\Psi_{44}(\mathbf{N}_0\otimes\mathbf{N}_0) + \Psi_{45}(\mathbf{N}_0\otimes\mathbf{N}_{0\mathbf{C}} + \mathbf{N}_{0\mathbf{C}}\otimes\mathbf{N}_0) + \Psi_{55}\mathbf{N}_{0\mathbf{C}}\otimes\mathbf{N}_{0\mathbf{C}}] \end{aligned} \tag{25}$$

$$\begin{aligned}\bar{\mathbf{A}}^{m}_{F_2m} := {} & 4[(\Psi_{16} + I_1\Psi_{26} + \Psi_7)(\mathbf{1}\otimes\mathbf{M}_0 + \mathbf{M}_0\otimes\mathbf{1}) + (\Psi_{17} + I_1\Psi_{27})(\mathbf{1}\otimes\mathbf{M}_{0\mathbf{C}} + \mathbf{M}_{0\mathbf{C}}\otimes\mathbf{1})] \\ & + 4[-\Psi_{26}(\mathbf{C}\otimes\mathbf{M}_0 + \mathbf{M}_0\otimes\mathbf{C}) - \Psi_{27}(\mathbf{C}\otimes\mathbf{M}_{0\mathbf{C}} + \mathbf{M}_{0\mathbf{C}}\otimes\mathbf{C})] \\ & + 4[\Psi_{66}(\mathbf{M}_0\otimes\mathbf{M}_0) + \Psi_{67}(\mathbf{M}_0\otimes\mathbf{M}_{0\mathbf{C}} + \mathbf{M}_{0\mathbf{C}}\otimes\mathbf{M}_0) + \Psi_{77}\mathbf{M}_{0\mathbf{C}}\otimes\mathbf{M}_{0\mathbf{C}}] \end{aligned} \tag{26}$$

By forming a complex network surrounded by and, at the same time, entrapping water, proteoglycans, glycoproteins and elastin, collagen fibers can play a central role in the overall compressibility of the material by interacting with the isotropic matrix. When stretched, collagen fibers squeeze the surrounding interfibrillar matrix and have their diameters reduced. These coupled interactions can be included in the formulation by defining bilinear functions of I_3 and I_4, or I_3 and I_5, or I_3 and I_6, or I_3 and I_7, exhibiting the possibility to have non zero second derivatives appearing in the elasticity tensor (Eqs. (27) and (28)):

$$\tilde{\mathbf{A}}^{m}_{F_1 m} := 4\left[I_3 \Psi_{43}\left(\mathbf{C}^{-1} \otimes \mathbf{N}_0 + \mathbf{N}_0 \otimes \mathbf{C}^{-1}\right) + I_3 \Psi_{53}\left(\mathbf{C}^{-1} \otimes \mathbf{N}_{0\mathbf{C}} + \mathbf{N}_{0\mathbf{C}} \otimes \mathbf{C}^{-1}\right)\right] \tag{27}$$

$$\tilde{\mathbf{A}}^{m}_{F_2 m} := 4\left[I_3 \Psi_{63}\left(\mathbf{C}^{-1} \otimes \mathbf{M}_0 + \mathbf{M}_0 \otimes \mathbf{C}^{-1}\right) + I_3 \Psi_{73}\left(\mathbf{C}^{-1} \otimes \mathbf{M}_{0\mathbf{C}} + \mathbf{M}_{0\mathbf{C}} \otimes \mathbf{C}^{-1}\right)\right] \tag{28}$$

The possible coupling between the tensorial invariants related to the fibers and that related to the volume ratio ($J = \sqrt{I_3}$) can help to characterize typical behaviors such as the fact that fluid exudation in soft tissues can be observed in particular directions (Armstrong et al., 1984). The fluid exudation that affects the global compressibility is probably (this remains to be proven) channeled by the fiber network. It was shown that water contributes significantly to the nonlinear viscoelastic behavior of ligaments (Chimich et al., 1992). The water content may play a role in conditioning the distance at which collagen fibers can interact mechanically or biochemically. As proteoglycans are highly hydrophilic molecules, significant pressure gradients are likely to be generated within the tissue and hence producing redistribution of the water.

Several experimental studies have reported very interesting observations, namely the fact that the stiffness of connective soft tissues is much higher than that of a single collagen fiber (Minns and Soden, 1973; Hayashi et al., 2000).

In addition to evident size effects as mentioned by the previous authors, obvious explanations could be put forward by considering that mechanical interactions between the ground substance and the collagen fibers play a major role in the significant difference of stiffness observed (Hayashi et al., 2000). At the atomic level, covalent liaisons between the components of the ground substance and the collagen fibers are certainly responsible for a strengthening of the whole structure. Collagen fibers, bonded together by an interfibrillar matrix, are arranged in bundles which in turn are structured in fascicles. This arrangement is likely to produce shear between the collagen fibers and the matrix when a ligament or a tendon is loaded in tension. This effect can explain the stronger stiffness of the connective tissue over a single collagen fiber. Others reasons include the presence of elastin fibers that provide the elastic recovery capabilities (storage of elastic energy) of a ligament and that are responsible for bringing back collagen fibers in their crimped state. Their actions could be viewed as a resisting factor in the elongation of the collagen fibers. In connection with this, non-uniform pre-stretch of the collagen fibers is probably present in an apparent relaxed soft connective tissue due to its strain and stress history.

Rate effects such as viscoelasticity may be accountable for the experimental observations mentioned above (although they are supposed to be performed on conditioned specimens) because in an isolated collagen fiber, viscosity interactions provided by the presence of the ground substance (water, glycoproteins, elastin) are missing and thus alter the apparent stiffness. Some ligaments are encapsulated in a membrane, called epiligament (Frank and Shrive, 1999), which contains randomly orientated collagen fibrils and a network of blood vessels branching and penetrating the intrasubstance of the ligament, running along and between the collagen fascicles. This is the kind of structural arrangement that can add significant stiffness to a connective tissue even thought its structural components have lower stiffnesses.

The combined interactions of the two families of fibers with the isotropic matrix are governed by $\bar{\mathbf{A}}^{m}_{F_1 F_2 m}$ and $\tilde{\mathbf{A}}^{m}_{F_1 F_2 m}$. $\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}$ is a second-order tensor that reflects the relative orientation of the two family of fibers and which annihilates if the orientation is orthogonal. It is well known that fibers orient themselves according to the load they carry. If we consider a ligament in a relaxed state, at each local continuum location considered, $\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}$ is probably nonzero, but as the ligament is submitted to multiaxial loading, fibers may align in a way such that $\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}$ becomes zero and thus modifying locally the stiffness and producing singularities in the stress distribution. These phenomena could account partly for injuries when, in addition to high strain rates, ligaments are loaded in abnormal directions.

$$\bar{\mathbf{A}}^{m}_{F_1F_2m} := 4\left[(\Psi_1 + I_1\Psi_{28})\left(\mathbf{1}\otimes\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0} + \mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\mathbf{1}\right) - \Psi_{28}\left(\mathbf{C}\otimes\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0} + \mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\mathbf{C}\right)\right] \tag{29}$$

$$\tilde{\mathbf{A}}^{m}_{F_1F_2m} := 4I_3\Psi_{38}\left(\mathbf{C}^{-1}\otimes\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0} + \mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\mathbf{C}^{-1}\right) \tag{30}$$

$\mathbf{A}^{m}_{F_1F_2}$ exhibits the pure mutual interaction between the two families of fibers. Stretch of a family can produce stress in the other family of fibers under various forms such as shear, compression or tension if they are intermeshed to each other. However, the difficulty of the experimental measurement of these effects is a real practical limitation for now, in addition to the very demanding requirements of testing of biological tissues.

$$\begin{aligned}\mathbf{A}^{m}_{F_1F_2} :=\; & 4[\Psi_{46}(\mathbf{N}_0\otimes\mathbf{M}_0 + \mathbf{M}_0\otimes\mathbf{N}_0) - \Psi_{47}(\mathbf{N}_0\otimes\mathbf{M}_{0\mathbf{C}} + \mathbf{M}_{0\mathbf{C}}\otimes\mathbf{N}_0)] \\ & + 4[\Psi_{56}(\mathbf{N}_{0\mathbf{C}}\otimes\mathbf{M}_0 + \mathbf{M}_0\otimes\mathbf{N}_{0\mathbf{C}}) - \Psi_{57}(\mathbf{N}_{0\mathbf{C}}\otimes\mathbf{M}_{0\mathbf{C}} + \mathbf{M}_{0\mathbf{C}}\otimes\mathbf{N}_{0\mathbf{C}})] \\ & + 4\left[\Psi_{48}\left(\mathbf{N}_0\otimes\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0} + \mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\mathbf{N}_0\right) + \Psi_{58}\left(\mathbf{N}_{0\mathbf{C}}\otimes\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0} + \mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\mathbf{N}_{0\mathbf{C}}\right)\right] \\ & + 4\left[\Psi_{68}\left(\mathbf{M}_0\otimes\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0} + \mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\mathbf{M}_0\right) + \Psi_{78}\left(\mathbf{M}_{0\mathbf{C}}\otimes\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0} + \mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\mathbf{M}_{0\mathbf{C}}\right)\right] \\ & + 4\left[\Psi_{88}\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\otimes\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0}\right]\end{aligned} \tag{31}$$

To the best of our knowledge, the expressions of the various terms of the elasticity tensor, containing coupling contributions between the matrix and the two families of fibers, have been not previously reported in the literature. The explicit dependence of the elasticity tensor on the partial derivatives of the strain energy function can serve as a basis to derive in a straightforward manner the elasticity tensor for a particular strain energy function. This is helpful to investigate particular mechanical effects determined by carefully chosen strain energy functions.

4.2. Elongation moduli in the material description

From the expression of the elasticity tensor one can define elongation moduli $\kappa_{\mathbf{n}_0}$ and $\kappa_{\mathbf{m}_0}$, respectively associated with the fiber directions $\mathbf{n}_0$ and $\mathbf{m}_0$. They characterize the stress response associated with the deformations in the fiber directions and are therefore directly related to the appropriate structural tensors by the following relationships:

$$\kappa_{\mathbf{n}_0} = \mathbf{N}_0 : (\mathbf{A}^{m}\cdot\mathbf{N}_0), \quad \kappa_{\mathbf{m}_0} = \mathbf{M}_0 : (\mathbf{A}^{m}\cdot\mathbf{M}_0) \tag{32}$$

It is worthy to note that these extension moduli, particularly $\kappa_{\mathbf{n}_0}$, when assessed in the linear phase (after recruitment of the collagen fibers) of the stress–strain curve (typically over 4% of strain) represent the classical Young's modulus reported in experimental studies considering ligament as simple elastic material (Butler et al., 1986).

4.3. Bulk modulus in the material description

The bulk modulus κ of the material can be defined as follows:

$$\kappa = \frac{1}{9}\mathbf{1} : (\mathbf{A}^{m}\cdot\mathbf{1}) \tag{33}$$

κ characterizes the volumetric stresses associated with volumetric deformations of the material. From (32) and (33), it is straightforward to derive the elastic moduli of the linear elasticity theory by assuming a state of vanishing strains. If the hypothesis of small perturbations is made and if it is assumed that fibers have no mechanical contribution at this strain regime, an equivalent isotropic bulk modulus can be deduced and used as a coefficient characterizing the initial compressibility of the material.

4.4. Elasticity tensor in the spatial configuration

The spatial counterpart of the material elasticity tensor, $\mathbf{A}^{\mathscr{S}}$, is defined by the push-forward relation:

$$(\mathbf{A}^{\mathscr{S}})_{ijkl} = \frac{1}{J} F_{iI} F_{jJ} F_{kK} F_{lL} A^{m}_{IJKL} \tag{34}$$

To avoid redundancy, the spatial elasticity tensor has not been split into various contributions as performed for the material elasticity tensor. The full general expression of $\mathbf{A}^{\mathscr{S}}$ is given in Eq. (35):

$$\begin{aligned}
\mathbf{A}^{\mathscr{S}} = {} & 4\left[(\Psi_{11} + 2I_1\Psi_{12} + \Psi_2 + I_1^2\Psi_{22})\mathbf{b}\otimes\mathbf{b} + \Psi_{22}\left(\mathbf{b}^2\otimes\mathbf{b}^2\right)\right] \\
& + 4\left[-(\Psi_{12} + I_1\Psi_{22})\left(\mathbf{b}\otimes\mathbf{b}^2 + \mathbf{b}^2\otimes\mathbf{b}\right) - \Psi_2\mathbf{I}_{\mathbf{b}^{-1}}\right] \\
& + 4\left[\left(I_3\Psi_3 + I_3^2\Psi_{33}\right)\mathbf{1}\otimes\mathbf{1} + I_3\Psi_3\mathbf{I} + I_3(\Psi_{13} + I_1\Psi_{23})(\mathbf{b}\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{b})\right] \\
& + 4\left[-I_3\Psi_{23}\left(\mathbf{b}^2\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{b}^2\right)\right] \\
& + 4[I_3I_4\Psi_{43}(\mathbf{1}\otimes\mathbf{N} + \mathbf{N}\otimes\mathbf{1}) + I_3I_4\Psi_{53}(\mathbf{1}\otimes\mathbf{N_b} + \mathbf{N_b}\otimes\mathbf{1})] \\
& + 4[I_3I_6\Psi_{63}(\mathbf{1}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{1}) + I_3I_6\Psi_{73}(\mathbf{1}\otimes\mathbf{M_b} + \mathbf{M_b}\otimes\mathbf{1})] \\
& + 4[I_4(\Psi_{14} + I_1\Psi_{24} + \Psi_5)(\mathbf{b}\otimes\mathbf{N} + \mathbf{N}\otimes\mathbf{b}) + I_4(\Psi_{15} + I_1\Psi_{25})(\mathbf{b}\otimes\mathbf{N_b} + \mathbf{N_b}\otimes\mathbf{b})] \\
& + 4\left[-I_4\Psi_{24}\left(\mathbf{b}^2\otimes\mathbf{N} + \mathbf{N}\otimes\mathbf{b}^2\right) - I_4\Psi_{25}\left(\mathbf{b}^2\otimes\mathbf{N_b} + \mathbf{N_b}\otimes\mathbf{b}^2\right)\right] \\
& + 4\left[I_4^2\Psi_{44}(\mathbf{N}\otimes\mathbf{N}) - I_4^2\Psi_{45}(\mathbf{N}\otimes\mathbf{N_b} + \mathbf{N_b}\otimes\mathbf{N}) + I_4^2\Psi_{55}\mathbf{N_b}\otimes\mathbf{N_b}\right] \\
& + 4[I_6(\Psi_{16} + I_1\Psi_{26} + \Psi_7)(\mathbf{b}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{b}) + I_6(\Psi_{17} + I_1\Psi_{27})(\mathbf{b}\otimes\mathbf{M_b} + \mathbf{M_b}\otimes\mathbf{b})] \\
& + 4\left[-I_6\Psi_{26}\left(\mathbf{b}^2\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{b}^2\right) - I_6\Psi_{27}\left(\mathbf{b}^2\otimes\mathbf{M_b} + \mathbf{M_b}\otimes\mathbf{b}^2\right)\right] \\
& + 4\left[I_6^2\Psi_{66}(\mathbf{M}\otimes\mathbf{M}) - I_6^2\Psi_{67}(\mathbf{M}\otimes\mathbf{M_b} + \mathbf{M_b}\otimes\mathbf{M}) + I_6^2\Psi_{77}\mathbf{M_b}\otimes\mathbf{M_b}\right] \\
& + 4[I_4I_6\Psi_{46}(\mathbf{N}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{N}) - I_4I_6\Psi_{47}(\mathbf{N}\otimes\mathbf{M_b} + \mathbf{M_b}\otimes\mathbf{N})] \\
& + 4[I_4I_6\Psi_{56}(\mathbf{N_b}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{N_b}) - I_4I_6\Psi_{57}(\mathbf{N_b}\otimes\mathbf{M_b} + \mathbf{M_b}\otimes\mathbf{N_b})] \\
& + 4\left[\sqrt{I_4I_6}(\Psi_1 + I_1\Psi_{28})(\mathbf{b}\otimes\mathbf{T_{nm}} + \mathbf{T_{nm}}\otimes\mathbf{b}) - \sqrt{I_4I_6}\Psi_{28}\left(\mathbf{b}^2\otimes\mathbf{T_{nm}} + \mathbf{T_{nm}}\otimes\mathbf{b}^2\right)\right] \\
& + 4\left[\sqrt{I_4I_6}I_3\Psi_{38}(\mathbf{1}\otimes\mathbf{T_{nm}} + \mathbf{1}\otimes\mathbf{T_{nm}})\right] \\
& + 4\left[\Psi_{48}\sqrt{I_4I_6}I_4(\mathbf{N}\otimes\mathbf{T_{nm}} + \mathbf{T_{nm}}\otimes\mathbf{N}) + \Psi_{58}\sqrt{I_4I_6}I_4(\mathbf{N_b}\otimes\mathbf{T_{nm}} + \mathbf{T_{nm}}\otimes\mathbf{N_b})\right] \\
& + 4\left[\Psi_{68}\sqrt{I_4I_6}I_6(\mathbf{M}\otimes\mathbf{T_{nm}} + \mathbf{T_{nm}}\otimes\mathbf{M}) + \Psi_{78}\sqrt{I_4I_6}I_6(\mathbf{M_b}\otimes\mathbf{T_{nm}} + \mathbf{T_{nm}}\otimes\mathbf{M_b})\right] \\
& + 4[\Psi_{88}I_4I_6\mathbf{T_{nm}}\otimes\mathbf{T_{nm}}]
\end{aligned} \tag{35}$$

where

$$(\mathbf{I}_{\mathbf{b}^{-1}})_{ijkl} := (\varphi_*\mathbf{I})_{ijkl} = -\frac{1}{2}\left(b_{ik}^{-1}b_{jl}^{-1} + b_{il}^{-1}b_{jk}^{-1}\right) \tag{36}$$

It appears that such a closed-form expression of the spatial version of the elasticity has never been reported in literature.

5. Mechanical symmetries for fiber-reinforced composites

The previous expressions of the stress and elasticity tensors have been established in the general case of a fiber-reinforced composite material containing two distinct families of fibers F_1 and F_2. No assumption was made regarding the mutual orientation of the two families of fibers. Orthotropic, transversely isotropic and isotropic symmetries are special cases that will be derived, in the next section, in a straightforward manner from the general anisotropic formulation. For sake of illustration, the degenerated expressions of the spatial elasticity tensors are presented for each material symmetry.

5.1. Orthotropic symmetry and locally orthotropic symmetry

When the principal preferred directions of the two families of fibers are mutually orthogonal in the reference configuration, the composite material possesses an *orthotropic symmetry* because there exist three orthogonal planes of symmetry: two normal to the fiber directions and one parallel to the surface where the fibers lie. This symmetry group requires nine independent scalar coefficients to fully characterize the material. The scalar product of the two unit vector $\mathbf{n}_0$ and $\mathbf{m}_0$ is zero, as the eighth invariant I_8 and its derivative with respect to $\mathbf{C}$. $\mathbf{A}^{m}$ and $\mathbf{S}$ are expressed by means of Eqs. (37) and (38) respectively.

$$\mathbf{A}^{m} := \bar{\mathbf{A}}^{m}_{m} + \tilde{\mathbf{A}}^{m}_{m} + \bar{\mathbf{A}}^{m}_{F_1m} + \tilde{\mathbf{A}}^{m}_{F_1m} + \bar{\mathbf{A}}^{m}_{F_2m} + \tilde{\mathbf{A}}^{m}_{F_2m} + \mathbf{A}^{m}_{F_1F_2} \tag{37}$$

$$\mathbf{S} = 2\left[(\Psi_1 + I_1\Psi_2)\mathbf{1} - \Psi_2\mathbf{C} + I_3\Psi_3C^{-1} + \Psi_4\mathbf{N}_0 + \Psi_5\mathbf{N}_{0\mathbf{C}} + \Psi_6\mathbf{M}_0 + \Psi_7\mathbf{M}_{0\mathbf{C}}\right] \tag{38}$$

One can define a *local orthotropy* when the two families of fibers are mechanically equivalent, i.e. it is possible to interchange $\mathbf{n}_0$ and $\mathbf{m}_0$ without affecting the properties of symmetry. In this case, the material is said to be *locally orthotropic* with respect to the mutually orthogonal planes which bisect the two families of fibers (with direction $\mathbf{n}_0$ and $\mathbf{m}_0$) and the plane in which the fibers lie. The dependence of $\mathbf{n}_0$ and $\mathbf{m}_0$ on Ψ is symmetric with respect to swap between $\mathbf{n}_0$ and $\mathbf{m}_0$. In this case, Ψ can be defined by means of I_1, I_2, I_3, I_8 and three additional invariants I_9, I_{10} and I_{11} defined as follows (Spencer, 1992):

$$I_9 = I_4 + I_6, \quad I_{10} = I_4I_6, \quad I_{11} = I_5 + I_7 \tag{39}$$

The second Piola–Kirchhoff stress tensor is defined as

$$\mathbf{S} = 2\left[(\Psi_1 + I_1\Psi_2)\mathbf{1} - \Psi_2\mathbf{C} + I_3\Psi_3\mathbf{C}^{-1} + \Psi_8\mathbf{T}_{\mathbf{n}_0\mathbf{m}_0} + \Psi_9\mathbf{Z}_0 + \Psi_{10}\mathbf{Y}_{\mathbf{n}_0\mathbf{m}_0} + \Psi_{11}\mathbf{Z}_{0\mathbf{C}}\right] \tag{40}$$

where

$$\mathbf{Z}_0 := \mathbf{N}_0 + \mathbf{M}_0, \quad \mathbf{Z}_{0\mathbf{C}} := \mathbf{N}_{0\mathbf{C}} + \mathbf{M}_{0\mathbf{C}}, \quad \mathbf{Y}_{\mathbf{n}_0\mathbf{m}_0} := I_6\mathbf{N}_0 + I_4\mathbf{M}_0 \tag{41}$$

To keep the present developments concise, the expression of the material elasticity tensor corresponding to local orthotropy is not presented.

5.2. Transversely isotropic symmetry

In this case, the material is assumed to be a solid isotropic matrix reinforced by a single family of fibers characterized by a fiber direction given by $\mathbf{n}_0$ (Fig. 2).

Five independent scalar coefficients are needed to define the constitutive law. Of course, I_6, I_7 and I_8 are no longer arguments of the strain energy function and the spatial elasticity tensor reduces to:

$$\mathbf{A}^{m} := \bar{\mathbf{A}}^{m}_{m} + \tilde{\mathbf{A}}^{m}_{m} + \bar{\mathbf{A}}^{m}_{F_1m} + \tilde{\mathbf{A}}^{m}_{F_1m} \tag{42}$$

This constitutive formulation was successfully used by Weiss et al. (1996)) to describe the mechanical behavior of fascia lata tendons and the collateral ligaments of the knee and was implemented into an implicit finite element code. This work brought a significant contribution to finite element modeling of

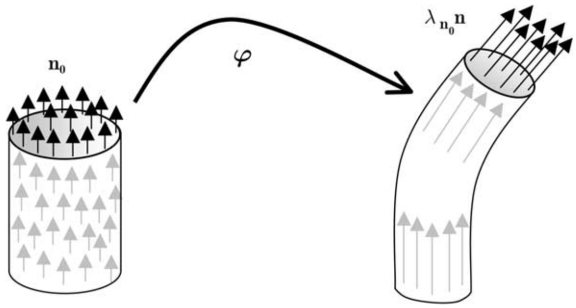

Fig. 2. Schematic representation of the deformation of a material made of an isotropic matrix reinforced by one family of fibers. $\mathbf{n}_0$ is the vector field carrying the preferred fiber direction in the reference configuration whilst $\mathbf{n}$ is its counterpart in the distorted configuration. Upon deformation $\mathbf{n}_0$ is mapped into $\lambda_{\mathbf{n}_0}\mathbf{n}$ by mean of the relation $\mathbf{F}\mathbf{n}_0 = \lambda_{\mathbf{n}_0}\mathbf{n}$.

ligaments and tendons by taking into account, for the first time in a three-dimensional continuum model, the directional properties of ligaments. The experimental validation of the mechanical formulation demonstrated the relevance of using the continuum theory of fiber-reinforced composites at finite strain.

5.3. Isotropic symmetry

The material is a solid isotropic matrix which is not reinforced by any family of fibers. The constitutive equations merely degenerate from the ones given for transversely isotropy by suppressing the terms involving I_4 and I_5. In this case, only two scalar coefficients are necessary to define the constitutive law and the elasticity tensor takes the simple following form:

$$\mathbf{A}^{m} := \bar{\mathbf{A}}^{m}_{m} + \tilde{\mathbf{A}}^{m}_{m} \tag{43}$$

Isotropic constitutive laws are widely used in rubber elasticity and the most popular ones are: the neo-Hookean model, the Mooney–Rivlin model, the Ogden power law or the Blatz–Ko model (Ogden, 1984). In the context of ligament and tendon modeling, Pioletti et al. (1998) used an isotropic law based on the strain energy function proposed by Veronda and Westmann (1970) for skin modeling. However, isotropic models were shown to lead to unrealistic results (Limbert and Taylor, 2001).

6. Kinematics constraints

In addition of being well justified for certain classes of problems, kinematics constraints can significantly simplify the formulation of the constitutive equations. Incompressibility and inextensibility in the fiber directions are examples of such constraints.

6.1. Incompressibility

The assumption of incompressibility is often made in finite elasticity because of the satisfactory results it gives (numerous materials behave nearly incompressible over a wide range of strains) and the simplification it brings in experimental measurements (allows to compute out-of-plane deformations, for example). This assumption increases considerably the difficulty in finite element analyses because of various numerical singularities generated (Flory, 1961). As mentioned earlier, in this case, $I_3 = 1$, all the derivatives of Ψ with respect to I_3 are zero and an arbitrary pressure p, determined only by the equations of equilibrium or motion and the boundary conditions (not the constitutive equations), enters the stress under the form of a Lagrange multiplier as a reaction to the kinematics constraint of incompressibility.

6.2. Inextensibility

In fiber-reinforced composite materials, the stiffness of the fibers is generally much stronger than that of the matrix. This implies that the extension moduli of the material in the fiber directions are much more larger that their shear moduli. The material will therefore be more likely to deform in a deformation mode other than extension in the fiber direction. This is a very relevant issue in soft tissue modeling as numerous studies have suggested and shown experimentally that a stressed fiber aligns to avoid a mechanical stimulus in the fiber direction under cyclic deformation (Yamada et al., 2000). When the material is inextensible in the two fiber directions $\mathbf{n}_0$ and/or $\mathbf{m}_0$, the inextensibility condition means that $I_4 = 1$ and/or $I_6 = 1$. As for the incompressible case, fiber reaction stresses enter the stresses as reactions to the kinematics constraints of inextensibility.

Incompressibility and inextensibility can coexist in the same constitutive law. Incompressibility is broadly used in finite element modeling of ligaments and tendons but inextensibility is only used in mathematical models describing ligaments as a collection of extensible and isometric fibers (O'Connor and Zavatsky, 1993). The hypothesis of inextensibility is probably relevant for finite element analyses of ligaments for particular fiber bundles within a ligament and for particular ranges of motion. Its influence should be assessed in order to allow possible simplification of constitutive laws.

7. Constitutive restrictions

Constitutive restrictions guiding the choice of the strain energy function have been largely investigated, especially for isotropic solids at finite strain (Ciarlet, 1988; Mardsen and Hughes, 1994; Oden and Reddy, 1978; Ogden, 1984; Truesdell and Noll, 1992). These restrictions can be divided into *mathematical* and *physical* restrictions. Mathematical restrictions are established in order to insure the existence and/or uniqueness of the solution of the initial/boundary value problem whilst physical restrictions impose constraints such that the material behaves in a physically acceptable manner, at least in the accessible experimental domain. It is worth to emphasize that objectivity and material symmetry discussed previously are also mathematical restrictions put on the strain energy function. The consideration of these restrictive conditions can prevent introduction of non-physical behavior or can put particular limits on the domain of validity of a constitutive law, at the formulation level. In the context of nonlinear finite element analyses, the restrictions imposed on the strain energy function can give confidence in the existence of a solution, or can explain particular results when, for example, the solution of a particular initial/boundary value problem is not unique. As there does not exist a general constitutive inequality encompassing all the required physical and mathematical properties, this leaves an open field for investigation.

8. Concluding remarks

The theory of fiber-reinforced composites developed by Spencer (1992) has been presented and extended in the context of the constitutive modeling of biological soft connective tissues. New closed-form expressions of the material and the spatial versions of the elasticity tensors have been derived for continuum fiber-reinforced composite material containing up to two families of fibers. The derivations have been performed without restricting the way the strain energy function depends on its arguments. The coupling terms appearing by successive differentiation of the strain energy function have been isolated and discussed in connection with the modeling of ligaments and tendons. These terms are representative of the mutual micromechanical interactions between the matrix and the two families of fibers. It was described how particular mechanical effects observed in biological structures can be accounted for by choosing appropriate functional forms of the strain energy function with respect to its arguments, that is tensorial invariants of the strain and structural tensor agencies.

The development of the expressions for the elasticity tensors in the material and spatial descriptions is of interest, because, in addition of their relevance to predict and explore the mechanical behavior of a given material, elasticity tensors hold fundamental mathematical properties of the constitutive law. Stability studies and constitutive restrictions generally rely on arguments based on these properties.

The general expressions of the stress and elasticity tensors are also essential in the finite element implementation of constitutive laws for fiber-reinforced composites and it is hoped that they will be useful with this regards. In implicit scheme based finite element methods, the elasticity tensor is used to calculate the tangent matrix which governs the convergence of the system of nonlinear equations whereas, in explicit analyses, the various coefficients of the elasticity tensors are used to calculate the largest stable time step by the mean of the equivalent Lamé's moduli.

The present phenomenological formulation is fairly simple but its drawback lies in the fact that the tensorial invariants of the right Cauchy–Green deformation tensor and agencies of structural tensors considered do not have all an easy physical interpretation like I_5 and I_7. Applicability of the general fiber-reinforced composite model remains to be explored on experimental grounds but with suitable experimental material characterization one can envisage to integrate complex interactions between elemental constituents within a constitutive law.

Although the developments presented in this paper are aimed at the constitutive modeling of biological soft connective tissues, there is no restriction in using them for the formulation of constitutive laws for other (biological) structures. Indeed, skin, intervertebral discs, arteries, among others, due to their fibrous structures and strong anisotropy are well suited candidates for the application of a continuum theory of fiber-reinforced composites. There is therefore plenty of room for investigative research work in the field of (continuum) fiber-reinforced composite materials would it be experimental, analytical or computational.

References

Amiel, D., Woo, S.L.Y., Harwood, F.L., Akeson, W.H., 1982. The effect of immobilization on collagen turnover in connective tissue. A biochemical–biomechanical correlation. Acta Orthopaedica Scandinavica 53, 325.

Armstrong, C.G., Lai, W.M., Mow, V.C., 1984. An analysis of the unconfined compression of articular cartilage. Journal of Biomechanical Engineering 106, 165–173.

Boehler, 1978. Lois de comportement anisotrope des milieux continus. Journal de Mécanique 17, 153–190.

Butler, D.L., Kay, M.D., Stouffer, D.C., 1986. Comparison of material properties in fascicle-bone units from human patellar tendon and knee ligaments. Journal of Biomechanics 19 (6), 425–432.

Chimich, D.D., Shrive, N.G., Marchuk, L., Bray, R.C., 1992. Water content alters the viscoelastic behaviour of the normal adolescent rabbit medial collateral ligament. Journal of Biomechanics 25 (18), 831–837.

Ciarlet, P., 1988. Mathematical Foundations of Elasticity. North-Holland Publishing Company, Amsterdam.

Cohen, H., Wang, C.C., 1987. On the response and symmetry of elastic materials with internal constraints. Archive of Rational Mechanics and Analysis 99, 1.

Coleman, B.D., Noll, W., 1964. Material symmetry and thermostatic inequalities in finite elastic deformations. Archive of Rational Mechanics and Analysis 17, 87–111.

Ericksen, J.L., 1978. On the symmetry and stability of thermoelastic solids. Journal of Applied Mechanics 45, 740.

Ericksen, J.L., 1979. On the symmetry of deformable crystals. Archive of Rational Mechanics and Analysis 72, 1–13.

Ericksen, J.L., Rivlin, R.S., 1954. Large elastic deformations of homogeneous anisotropic materials. Journal of Rational Mechanics and Analysis 3, 281–301.

Flory, P.J., 1961. Thermodynamic relations for high elastic materials. Transactions of the Faraday Society 57, 829–838.

Frank, C.B., Shrive, N.G., 1999. Biological materials—2.5 LIGAMENT. In: Biomechanics of the Musculo-skeletal System. Chichester, New York, pp. 107–126.

Fung, Y.C., 1981. Biomechanics: Mechanical Properties of Living Tissues. Springer-Verlag, New York.

Hahn, T., 1987. Space-group Symmetry, (A), second edition. Dreidel D, Dordrecht.

Hayashi, K., Miyazaki, H., Hasegawa, Y., 2000. Mechanical properties of cells and collagen fiber. Proceedings of the 10th International Conference on Biomedical Engineering, Singapore.

Hirokawa, S., Tsuruno, R., 2000. Three-dimensional deformation and stress distribution in an analytical/computational model of the anterior cruciate ligament. Journal of Biomechanics 33, 1069–1077.

Holzapfel, G.A., Eberlein, R., Wriggers, P., Weizsäcker, H.W., 1996. Large strain analysis of soft biological membranes: Formulation and finite element analysis. Computer Methods in Applied Mechanics and Engineering 132, 45–61.

Humphrey, J.D., 1990. Determination of a constitutive relation for passive myocardium. I. A new functional form. II. Parameter estimation. Journal of Biomechanical Engineering 112, 333–339, 340–346.

Humphrey, J.D., Yin, F.C.P., 1987. On constitutive relations and finite deformations of passive cardiac tissue. I. A pseudostrain energy function. Journal of Biomechanical Engineering 109, 298–304.

Kennedy, J.C., Hawkins, R.J., Willis, R.B., Danylchuck, K.D., 1976. Tension studies of human knee ligaments. Yield point, ultimate failure, and disruption of the cruciate and tibial collateral ligaments. Journal of Bone and Joint Surgery (Am) 58 (3), 350–355.

Limbert, G., Taylor, M., 2001. Three-dimensional finite element modelling of the human anterior cruciate ligament. Influence of the initial stress field. In: Computer Methods in Biomechanics and Biomedical Engineering, vol. 3. Gordon and Breach Science Publishers, New York, pp. 355–360.

Mardsen, J.E., Hughes, T.J.R., 1994. Mathematical Foundations of Elasticity. Dover, New York.

Minns, R.J., Soden, P.D., 1973. The role of the fibrous components and ground substance in the mechanical properties of biological tissues: a preliminary investigation. Journal of Biomechanics 6, 153–155.

Negahban, M., Wineman, A.S., 1989a. Material symmetry and the evolution of anisotropies in a simple material. I. Change of reference configuration. International Journal of Non-linear Mechanics 24, 521–536.

Negahban, M., Wineman, A.S., 1989b. Material symmetry and the evolution of anisotropies in a simple material. II. The evolution of material symmetry. International Journal of Non-linear Mechanics 24, 537–549.

O'Connor, J.J., Zavatsky, A., 1993. Anterior cruciate ligament forces in activity. In: The Anterior Cruciate Ligament: Current and Future Concepts. Raven Press, New York.

Oden, J.T., Reddy, J.N., 1978. An Introduction to the Mathematical Theory of Finite Elements. Academic Press, New York.

Ogden, R.W., 1984. Non-Linear Elastic Deformations. Ellis Horwood Ltd., West Sussex, England.

Pioletti, D.P., Rakotomanana, L., Benvenuti, J.F., Leyvraz, P.F., 1998. Finite element model of the human anterior cruciate ligament. In: Computer Methods in Biomechanics and Biomedical Engineering, vol. 2. Gordon and Breach Science Publishers, New York, pp. 561–568.

Spencer, A.J.M., 1992. Continuum Theory of the Mechanics of Fibre-reinforced Composites. Springer-Verlag, New York.

Thielke, R.J., Vanderby Jr., R., Grood, E.S., 1995. Volumetric changes in ligaments under tension. Proceedings of the ASME BED Conference 29, 197.

Truesdell, C., Noll, W., 1992. The Non-linear Field Theories of Mechanics. Springer-Verlag, New York.

Veronda, D.R., Westmann, R., 1970. Mechanical characterization of skin—finite deformations. Journal of Biomechanics 3, 111–124.

Weiss, J.A., Maker, B.N., Govindjee, S., 1996. Finite element implementation of incompressible transversely isotropic hyperelasticity. Computer Methods in Applied Mechanics and Engineering 135, 107–128.

White, A., Handler, P., Smith, E.L., 1964. Principles of Biochemistry. McGraw-Hill, New York.

Wineman, A.S., Pipkin, A.C., 1964. Material symmetry restrictions on constitutive equations. Archive of Rational Mechanics and Analysis 17, 184–214.

Woo, S.L.Y., Maynard, J., Butler, D.L., Lyon, R., Torzilli, P.A., Akeson, W.H., 1988. Ligament, tendon and joint capsule insertions into bone. In: Injury and Repair of the Musculoskeletal Soft Tissues, Park Ridge. American Academy of Orthopaedic Surgeons, Park Ridge, IL, pp. 133–166.

Yamada, H., Takemasa, T., Yamaguchi, T., 2000. Theoretical study of intracellular stress fiber orientation under cyclic deformation. Journal of Biomechanics 33, 1501–1505.

Zheng, Q.S., Boeheler, J.P., 1994. The description, classification, and reality of material and physical symmetries. Acta Mechanica 102, 73–89.